

# Metric Formulation of Ghost-Free Multivielbein Theory

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**ABSTRACT:** We formulate the recently proposed ghost-free theory of multiple interacting vielbeins in terms of their corresponding metrics. This is achieved by reintroducing all local Lorentz invariances broken by the multivielbein interaction potential which, in turn, allows us to explicitly separate the gauge degrees of freedom in the vielbeins from the components of the metrics by an appropriate gauge choice. We argue that the gauge choice does not spoil the no-ghost proof of the multivielbein theory, hence the multimetric theory is ghost-free. We further show the on-shell equivalence of the metric and vielbein descriptions, first in general and thereafter in two illustrative examples.

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## 1 Introduction and summary

Theories of massive and multiple interacting spin-2 fields are often discussed in terms of theories of massive gravity and bimetric or multimetric gravity. Until recently constructing theories of this type, that were at least classically consistent, had remained an unsolved problem. This was due to the presence of Boulware-Deser ghosts [1, 2] that generically arise in massive spin-2 theories. The situation drastically changed since the massive gravity proposal of [3, 4], that could describe a massive spin-2 field in flat spacetime. That this model avoided the BD ghost at the nonlinear level was proven in [5–7], which also extended it to a massive spin-2 field in curved spacetime.

Subsequently, a ghost-free theory of two interacting spin-2 fields was constructed in [8] as a bimetric theory (for earlier work on bimetric theories see [9–16]). As a generalization of the bimetric theory, in a remarkable recent work an interacting multivielbein theory of  $\mathcal{N}$  vielbeins was shown to be ghost-free [17]. For  $\mathcal{N} = 2$  it was possible to express the interactions in terms of metrics, reobtaining the bimetric theory. For earlier attempts at vielbein formulations see [18–21].

A difficulty encountered in [17] was in finding an expression for the multivielbein interactions in terms of metrics for  $\mathcal{N} > 2$ . If the multivielbein theory is regarded as a theory of interacting spin-2 fields, then finding such a metric description is desirable since in setups with general covariance, metrics provide a minimal description of spin-2 fields.

To summarize, in this paper we show that a gauge invariant generalization of the multivielbein theory can indeed be written as a ghost-free theory of  $\mathcal{N}$  interacting metrics. In [17], the authors consider a set of interaction terms involving  $\mathcal{N}$  vielbeins with a single local Lorentz invariance. We reintroduce the  $\mathcal{N} - 1$  broken local Lorentz invariances, now allowing for different gauge fixings, and argue that the no-ghost proof of [17] is still valid. This enables us to express the interaction potential in terms of  $\mathcal{N}$  metrics as well as  $\mathcal{N} - 1$  non-dynamical antisymmetric tensor fields. The latter are determined by their equations of motion which, as we demonstrate, are equivalent to the constraint equations that arise in the vielbein formulation.

## 2 Review of multivielbein theory

In this section we describe the multivielbein theory that was recently shown to be free of the Boulware-Deser ghosts [1, 2] in [17]. We also discuss the generality of the no-ghost proof of [17] which is essential for the consistency of the multimetric formulation of the theory.

*The action:* The multimetric theory is constructed in terms of  $\mathcal{N}$  vielbeins  $E^a{}_\mu(I)$ , with associated metrics,

$$g_{\mu\nu}(I) = E^a{}_\mu(I) \eta_{ab} E^b{}_\nu(I), \quad (2.1)$$

where  $I = 1, \dots, \mathcal{N}$ . The kinetic part of the action is a sum of Einstein-Hilbert actions for the individual metrics,

$$\sum_{I=1}^{\mathcal{N}} M_{\text{Pl}}^{d-2}(I) \int d^d x \sqrt{-\det g(I)} R(g(I)). \quad (2.2)$$

The interaction part of the action is the mass potential,

$$\frac{m^2}{4} \int d^d x U = \sum_{\{I\}} \frac{m^2}{4} \int d^d x T^{I_1 \dots I_d} U_{I_1 \dots I_d} \quad (2.3)$$

where the  $T^{I_1 \dots I_d}$  are constant coefficients of mass dimension  $d - 2$  and the functions  $U_{I_1 \dots I_d}$  are given entirely in terms of the vielbeins <sup>1</sup>,

$$U_{I_1 \dots I_d} = \tilde{\epsilon}^{\mu_1 \dots \mu_d} \tilde{\epsilon}_{a_1 \dots a_d} E_{\mu_1}^{a_1}(I_1) \dots E_{\mu_d}^{a_d}(I_d) \quad (2.4)$$

Here, both  $\tilde{\epsilon}^{\mu_1 \dots \mu_d}$  and  $\tilde{\epsilon}_{a_1 \dots a_d}$  are tensor densities which means that they are, effectively, invariant under general coordinate and Lorentz transformations respectively. Then (2.4) transforms in the same way as the usual volume form  $\sqrt{-\det g}$  and the action is invariant.

*Symmetries:* The Einstein-Hilbert actions (2.2) are invariant under independent local Lorentz transformations of the  $\mathcal{N}$  vielbeins  $E_{\mu}^a(I)$ . But in writing the interaction terms (2.4), the Lorentz frames of all vielbeins have been identified with each other and with that of  $\tilde{\epsilon}_{a_1 \dots a_d}$  so that these terms are invariant under a single local Lorentz group that acts on all vielbeins in the same way. The remaining  $\mathcal{N} - 1$  broken Lorentz groups have, in fact, been used to identify the frames to get to (2.4), as will become clearer in the next section. Hence, these  $\mathcal{N} - 1$  broken groups are no longer available to gauge away  $(\mathcal{N} - 1) \times d(d - 1)/2$  components of the vielbeins, or to rotate them to other frames. In this sense, the  $E_{\mu}^a(I)$  in (2.4) appear in given Lorentz frames, each with  $d^2$  independent components, modulo the overall local Lorentz transformations.

*Equations of motion:* In the multivielbein theory, these are  $\delta S / \delta E_{\mu}^a(I) = 0$ , that is  $d^2$  equations for each  $I$ . Derivatives of  $E_{\mu}^a(I)$  or  $g_{\mu\nu}(I)$  appear only through the Einstein tensor and hence are contained in the symmetric combinations,

$$\frac{\delta S}{\delta E_{\mu}^a(I)} E^c_{\mu}(I) \eta_{cb} + \frac{\delta S}{\delta E_{\mu}^b(I)} E^c_{\mu}(I) \eta_{ca} = 0. \quad (2.5)$$

The remaining antisymmetric combinations do not contain derivatives and thus give  $d(d - 1)/2$  constraints equations [17],

$$\frac{\delta U}{\delta E_{\mu}^a(I)} E^c_{\mu}(I) \eta_{cb} - \frac{\delta U}{\delta E_{\mu}^b(I)} E^c_{\mu}(I) \eta_{ca} = 0. \quad (2.6)$$

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<sup>1</sup>For  $\mathcal{N} = 2$ , this type of structure appeared in [18] in the context of supersymmetric bimetric gravity. Massive gravity in terms of vielbeins was also discussed in [19].

This is the correct number of equations to reduce the number of independent components of each vielbein from  $d^2$  to  $d(d+1)/2$ , equal to the number of independent components of the corresponding metric. In practice, non-trivial equations (2.6) exist only for  $\mathcal{N} - 1$  of the vielbeins (due to the overall Lorentz invariance of the action), while the last vielbein can be reduced to  $d(d+1)/2$  components using this overall local Lorentz invariance [17]. The surviving  $\mathcal{N}d(d+1)/2$  components are then governed by (2.5).

*Attempt at a multimetric description:* Based on the above discussion, it is natural to conjecture that solving the constraint equations (2.6) may hopefully lead to expressions for the vielbeins in terms of the metrics and thus enable us to express the multivielbein interactions (2.4) as multimetric interactions. However, this scheme is difficult to realize beyond  $\mathcal{N} = 2$ . Indeed in [17] it was found that for  $\mathcal{N} = 2$ , the constraints (2.4) are solved by a certain condition on the two vielbeins that also expresses them in terms of two metrics, thus recovering the bimetric theory of [8]. However, [17] also found that for  $\mathcal{N} > 2$  (except for cases that are trivial extensions of  $\mathcal{N} = 2$ ), the constraints (2.4) depend on the free parameters of the theory and it is not easy to find solutions that lead to generic expressions for the vielbeins in terms of the metrics.

While in many ways, working with the multivielbein theory may be easier than working with the corresponding multimetric theory, the motivation for expressing the multivielbein action in terms metrics is the final aim of using this as a theory of interacting spin-2 fields. In setups with general covariance, we are familiar with describing spin-2 fields in terms of symmetric rank-2 tensors. In this sense, spin-2 fields are more closely related to metrics than to vielbeins where one has to solve more equations to get to the spin-2 content. Hence, as a theory of spin-2 fields, it is desirable that the multivielbein interactions are also expressible in terms of the metrics, or rank-2 symmetric tensors in general. An expression for the multivielbein interactions in terms of metrics is obtained in the next section.

*Generality of the no-ghost proof:* In [17], Hinterbichler and Rosen showed that the above multivielbein theory has the right number of constraints to eliminate the Boulware-Deser ghosts. Here we review the main aspect of the ghost analysis, in particular emphasizing that the no-ghost proof of [17] is more general than the setup considered there explicitly. This is required for the consistency of the multimetric formulation found in the next section.

To analyze the ghost content at the nonlinear level one has to work in the ADM formulation and write the metrics  $g_{\mu\nu}(I)$  in a  $1 + (d-1)$  decomposition in terms of the lapses  $N(I)$ , shifts  $N_i(I)$  and spatial metrics  $g_{ij}(I)$  [22]. Such metrics are obtained from Lorentz transformations of constrained vielbeins and [17] chooses the parameterization,

$$E^a_\mu(I) = \hat{\Lambda}^a_b(\alpha_r; I) \hat{E}^a_\mu, \quad \hat{E}^a_\mu = \begin{pmatrix} N(I) & 0 \\ e^{\hat{b}}_j(I) N^j(I) & e^{\hat{b}}_i(I) \end{pmatrix}. \quad (2.7)$$

$E^a_\mu$  has  $d^2$  parameters: the  $\hat{\Lambda}$ , satisfying  $\hat{\Lambda}^T \eta \hat{\Lambda} = \eta$ , depend on  $d(d-1)/2$  parameters  $\alpha_r$  and, for definiteness, we take the constrained vielbeins  $\hat{E}$  to depend only on the  $d(d+1)/2$  parameters that also appear in the metric.

The no-ghost proof [17] rests on the possibility that all  $E(I)$  can be written as in (2.7) with Lorentz matrices  $\hat{\Lambda}(\alpha_r; I)$  that do not depend on the  $N(I)$  and  $N_i(I)$ . Then only the first column of  $E^a_\mu$  contains linear combinations of  $N$  and  $N_i$  and, due to the antisymmetry of (2.4), the potential  $U$  is linear in these variables. Hence,  $N(I)$  and  $N_i(I)$  are Lagrange multipliers and their equations of motion contain the primary constraints that eliminate the ghosts (secondary constraints that remove the momentum conjugate to the ghost are also expected to arise as in [7, 23]).

To obtain a multimetric form for (2.4), one needs to go through vielbeins that are more general than (2.7). In the ADM formalism, a generic vielbein can always be put in the form

$$E^a_\mu(I) = \tilde{\Lambda}^a_b(\hat{E}, \alpha_r; I) \hat{E}^a_\mu(I), \quad (2.8)$$

where  $\hat{E}$  has the same form as in (2.7) with  $d(d+1)/2$  parameters, but now the Lorentz matrix  $\tilde{\Lambda}$  may depend on the parameters of  $\hat{E}$  (including the  $N$  and  $N_i$ ) along with  $d(d-1)/2$  other independent parameters  $\alpha_r$ . This cannot be converted to the form (2.7) by available symmetries and hence does not satisfy the no-ghost criterion straightforwardly.

However, it is easy to argue that the no-ghost proof of [17] also applies to (2.8) as is best seen in the Cayley parameterization of the Lorentz matrix  $\tilde{\Lambda}$  (see equation (4.2)). This makes it very explicit that the  $d^2$  components of  $\tilde{\Lambda}$  depend on the components of  $\hat{E}$  and the  $\alpha_r$  only through  $d(d-1)/2$  functions  $\tilde{A}_r$  (which in this parameterization are arranged as components of an antisymmetric tensor  $\tilde{A}_{ab}$ ). Explicitly (suppressing the index  $I$ ),

$$\tilde{\Lambda}^a_b = \tilde{\Lambda}^a_b(\tilde{A}_r(\hat{E}, \alpha)) \quad (2.9)$$

Since  $E^a_\mu$  depends on  $d^2$  independent parameters, the relation between the  $\tilde{A}_r$  and the functions  $\alpha_r$  is invertible. This insures that if we now make a field redefinition from the fields  $\alpha_r$  to the fields  $\tilde{A}_r$  and treat the new fields as independent of  $N$  and  $N_i$ , then the set of  $(\hat{E}, \alpha_r)$  equations of motion are equivalent to the set of  $(\hat{E}, \tilde{A}_r)$  equations of motion, as can be explicitly checked.<sup>2</sup> In terms of the redefined fields, the  $E^a_\mu$  of (2.8) satisfy the same no-ghost criteria as vielbeins in (2.7) and the no-ghost proof of [17] still applies. This fact is important for the absence of ghosts in the multimetric form of the action obtained in the next section.

### 3 Multimetric action from the multivielbein theory

Here we obtain a metric expression for the multivielbein interactions (2.4) before solving the constraints (2.6). The difficulty encountered earlier is avoided by going away from the gauge fixed form of (2.4) and the restricted choice of parameterization.

*Completely gauge invariant multivielbein action:* In principle, when the  $\mathcal{N}$  metrics  $g_{\mu\nu}(I)$  are written in terms of vielbeins,

$$g_{\mu\nu}(I) = e^a_\mu(I) \eta_{ab}(I) e^b_\nu(I), \quad (3.1)$$

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<sup>2</sup>This is the analogue of “field redefinitions” encountered in the no-ghost proofs of massive gravity and bimetric gravity [4–6, 8]

there is no reason to identify the  $\mathcal{N}$  Lorentz frames with each other and with the Lorentz frame of  $\tilde{\epsilon}_{a_1 \dots a_d}$ . To emphasize this, we denote the vielbeins in the independent Lorentz frames by  $e^a_\mu(I)$ , in contrast to the  $E^a_\mu(I)$  of the previous section. Then, in general, the Lorentz frames of the  $e^a_\mu(I)$  are related to that of  $\tilde{\epsilon}_{a_1 \dots a_d}$  by local Lorentz transformations (LLT)  $\Lambda^a_b(I)$ ,

$$\Lambda^a_b(I) \eta^{bd}(I) \Lambda^c_d(I) = \eta^{ac}(I), \quad (3.2)$$

and the interaction term preserving all  $\mathcal{N}$  local Lorentz invariances is given by,

$$U_{I_1 \dots I_d} = \tilde{\epsilon}^{\mu_1 \dots \mu_d} \tilde{\epsilon}_{a_1 \dots a_d} \left[ \Lambda^{a_1}_{b_1}(I_1) e^{b_1}_{\mu_1}(I_1) \right] \dots \left[ \Lambda^{a_d}_{b_d}(I_d) e^{b_d}_{\mu_d}(I_d) \right]. \quad (3.3)$$

The  $\Lambda^{a_i}_{b_i}(I_i)$  are Stückelberg fields that restore the  $\mathcal{N} - 1$  broken LLT's of (2.4) and transform as bi-fundamentals. The  $a_i$  index transforms under LLT of the  $\tilde{\epsilon}_{a_1 \dots a_d}$  frame, while the  $b_i$  index transforms under the LLT of the corresponding  $e^{b_i}_{\mu_i}(I_i)$  frame.

The relation to the gauge fixed form of the interaction is easy to see. If we choose local Lorentz gauges that set all  $\Lambda^a_b(I)$  equal to each other then, since  $\det \Lambda(I) = 1$ , the interactions (3.3) reduce to (2.4), breaking the symmetry to the diagonal subgroup of the  $\mathcal{N}$  LLT's. However, for generic starting  $\Lambda(I)$ , the vielbeins in this gauge are now in the form (2.8) rather than (2.7). But, as discussed in the previous section, the no-ghost proof of [17] readily extends to this case and the completely gauge invariant form of multivielbein theory is ghost free for general  $\Lambda(I)$ . To find an expression for the multivielbein theory in terms of the metrics, one needs to choose a different gauge.

*Multimetric formulation:* Now we consider the multivielbein interactions in the form (3.3). Without loss of generality, let us pick out one of the vielbeins, say  $e^a_\mu(1)$  and express the volume element in terms of the corresponding metric  $g_{\mu\nu}(1)$  (3.1). To do this, we first have to identify the Lorentz frames of  $e^a_\mu(1)$  and  $\tilde{\epsilon}_{a_1 \dots a_d}$ . Since  $\tilde{\epsilon}_{a_1 \dots a_d}$  is Lorentz invariant, we have,

$$\tilde{\epsilon}_{a_1 \dots a_d} = \tilde{\epsilon}_{c_1 \dots c_d} (\Lambda^{-1})^{c_1}_{a_1}(1) \dots (\Lambda^{-1})^{c_d}_{a_d}(1). \quad (3.4)$$

Using this in (3.3) gives,

$$\tilde{\epsilon}^{\mu_1 \dots \mu_d} \tilde{\epsilon}_{c_1 \dots c_d} \left[ \Lambda^{c_1}_{b_1}(1, I_1) e^{b_1}_{\mu_1}(I_1) \right] \dots \left[ \Lambda^{c_d}_{b_d}(1, I_d) e^{b_d}_{\mu_d}(I_d) \right], \quad (3.5)$$

where we have defined,

$$\Lambda^c_b(1, I) = (\Lambda^{-1})^c_a(1) \Lambda^a_b(I). \quad (3.6)$$

Obviously,  $\Lambda^c_b(1, 1) = \delta^c_b$ . A generic  $\Lambda^c_b(1, I)$  transforms in a bi-fundamental, the upper index  $c$  transforming under the LLT of  $e^c_\mu(1)$  while the lower index  $b$  transforms under LLT of  $e^b_\mu(I)$ . In this form, the action still has all the  $\mathcal{N}$  local Lorentz invariances. Now we can introduce the  $g(1)$  volume element in (3.5) through the identity,

$$\tilde{\epsilon}_{c_1 \dots c_d} = \sqrt{-\det g(1)} \tilde{\epsilon}_{\nu_1 \dots \nu_d} e^{\nu_1}_{c_1}(1) \dots e^{\nu_d}_{c_d}(1), \quad (3.7)$$

that follows from the definition of the determinant.

To write the resulting expression in terms of metrics, consider (using the obvious matrix notation  $e = (e^a_\mu)$  for the vielbeins),

$$g^{-1}(1)g(I) = e^{-1}(1)\eta^{-1}e^{-1T}(1)e^T(I)\eta e(I). \quad (3.8)$$

Using LLT's,  $e(I)$  (for  $I = 2, \dots, \mathcal{N}$ ) can always be transformed to  $\bar{e}(I)$  such that the matrix  $\bar{e}(I)e^{-1}(1)\eta^{-1}$  is symmetric (this can be thought of as Lorentz transforming a polar decomposition of this matrix). This condition can also be expressed as,

$$\eta^{-1}e^{-1T}(1)\bar{e}^T(I) = [\eta^{-1}e^{-1T}(1)\bar{e}^T(I)]^T. \quad (3.9)$$

Then, in this local frame one has  $g^{-1}(1)g(I) = [e^{-1}(1)\bar{e}(I)]^2$  or,

$$\bar{e}^b_\mu(I) = e^b_{\lambda}(1) \left[ \sqrt{g^{-1}(1)g(I)} \right]^\lambda_\mu. \quad (3.10)$$

This also includes the trivial  $I = 1$  case. These gauge fixings of  $e(I)$ , for  $I = 2 \dots \mathcal{N}$ , identify their local Lorentz frames with that of  $e(1)$ . Now combining (3.5), (3.7), (3.10) and using the notation,

$$L^\nu_\lambda(I) \equiv e^\nu_a(1)\Lambda^a_b(1,I)e^b_\lambda(1), \quad (3.11)$$

we finally get an expression for the mass potential in terms of the  $\mathcal{N}$  metrics  $g(I)$ ,

$$U_{I_1 \dots I_d} = \sqrt{-\det g(1)} \tilde{\epsilon}^{\mu_1 \dots \mu_d} \tilde{\epsilon}_{\nu_1 \dots \nu_d} L^{\nu_1}_{\lambda_1}(I_1) \left[ \sqrt{g^{-1}(1)g(I_1)} \right]^{\lambda_1}_{\mu_1} \dots \\ \times L^{\nu_d}_{\lambda_d}(I_d) \left[ \sqrt{g^{-1}(1)g(I_d)} \right]^{\lambda_d}_{\mu_d}. \quad (3.12)$$

Thus the multimetric interaction is a direct generalization of the “deformed determinant” structure [24] and contains ingredients not more complicated than the matrix square-root first encountered in [4].

The mass potential (3.12) also contains the  $\mathcal{N} - 1$  matrices  $L(I)$  (with  $L^\mu_\nu(1) = \delta^\mu_\nu$ ). In  $d$  dimensions, each  $L^\nu_\lambda(I)$  (for  $I > 1$ ) inherits  $d(d-1)/2$  degrees of freedom from the local Lorentz transformation  $\Lambda^a_b(I)$ , so the pair  $(g_{\mu\nu}(I), L^\nu_\lambda(I))$  contains  $d^2$  degrees of freedom, the same as the content of the vielbein  $e^a_\mu(I)$ . Thus, even off-shell, the multimetric and multivielbein forms contain the same number of fields, after accounting for the local symmetries of the vielbein formalism. The  $d(d-1)/2$  independent components of each  $L(I)$  will be determined by their equations of motion as will be explained below. The use of  $L(I)$  makes it possible to express the mass potential as a finite polynomial in  $\sqrt{g^{-1}(1)g(I)}$ .

While the vielbein formulation is convenient for some applications, the metric formulation comes in handy when using the action to describe interactions of spin-2 fields which, in a general covariant setup, are described in terms of rank-2 symmetric tensors. To find the spin-2 content in the vielbein formulation one needs to eliminate  $d(d-1)/2$  a priori unknown combinations of the  $d^2$  components in each vielbein by the constraint equations (2.6), while the metric formulation manifestly isolates these from the symmetric tensors.

#### 4 The constraint equations for $L(I)$

The  $d(d-1)/2$  independent components of each  $L^\mu_\nu(I)$  appear as non-dynamical variables in the multimetric action and the associated equations of motion are constraints that could be solved to determine these. This is discussed below. For the sake of familiarity, the discussion is formulated in 4 dimensions but trivially extends to  $d$  dimensions.

*The  $L(I)$  equations:* Let us set  $g_{\mu\nu} = g_{\mu\nu}(1)$ ,  $e^a_\mu = e^a_\mu(1)$  and denote by  $L(I)$  the matrix with elements  $L^\mu_\nu(I)$ . It satisfies the property, inherited from (3.2),

$$L^T(I) g L(I) = g. \quad (4.1)$$

Hence  $L(I)$  contains the 6 independent degrees of freedom in  $\Lambda^a_b$  but it also depends on  $g_{\mu\nu}$ . The 6 independent components are determined through their equations of motion. To evaluate these, we first have to disentangle the Lorentz degrees of freedom in  $L(I)$  from its dependence on  $g_{\mu\nu}$ .

For this purpose we make use of the fact that Lorentz group can be parameterized in terms of antisymmetric matrices  $\hat{A}_{ab}$  as <sup>3</sup>

$$\Lambda^a_b = \left[ (\eta + \hat{A})^{-1} (\eta - \hat{A}) \right]^a_b, \quad \hat{A}_{ab} = -\hat{A}_{ba}. \quad (4.2)$$

This gives an expression for  $L^\mu_\nu(I)$  (3.11) in terms of  $A_{\mu\nu}(I) = e^a_\mu e^b_\nu \hat{A}_{ab}(I)$ ,

$$L^\mu_\nu(I) = \left[ (g + A(I))^{-1} (g - A(I)) \right]^\mu_\nu, \quad A_{\mu\nu}(I) = -A_{\nu\mu}(I). \quad (4.3)$$

Since the conditions on  $A_{\mu\nu}$  do not depend on the metric, the two can be varied independently and the 6  $A_{\mu\nu}$  equations of motion are the needed constraints. Varying  $A_{\mu\nu}$  one gets (suppressing the  $I$ ),

$$\delta L^\mu_\nu = -2 \left[ (g + A)^{-1} \delta A (g + A)^{-1} \right]^{\mu\lambda} g_{\lambda\nu}, \quad (4.4)$$

which, taking the antisymmetry of  $A$  into account, gives

$$\frac{\delta L^\mu_\nu}{\delta A_{\rho\sigma}} = \left[ (g + A)^{-1} \right]^{\mu\rho} \left[ (g + A)^{-1} \right]^{\sigma\lambda} g_{\lambda\nu} - \left[ (g + A)^{-1} \right]^{\mu\sigma} \left[ (g + A)^{-1} \right]^{\rho\lambda} g_{\lambda\nu}. \quad (4.5)$$

Since the multimetric action depends on the  $A(I)$  only through the  $L(I)$ , one can easily obtain the equations of motion (for each  $A(I)$  and  $L(I)$ ,  $I = 2, \dots, \mathcal{N}$ ),

$$\frac{\delta U}{\delta A_{\rho\sigma}} = \left( \left[ (g + A)^{-1} \right]^{\mu\rho} \left[ (g + A)^{-1} \right]^{\sigma\nu} - \left[ (g + A)^{-1} \right]^{\mu\sigma} \left[ (g + A)^{-1} \right]^{\rho\nu} \right) \frac{\delta U}{\delta L^\mu_\lambda} g_{\lambda\nu} = 0. \quad (4.6)$$

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<sup>3</sup>This is the Cayley transform of a Lorentz transformation. Here it is preferred over the exponential form  $e^w$ , since for a matrix  $w$ ,  $\partial(e^w)/\partial w^i_j$  does not have a closed form and is useful only perturbatively.



These are the  $6(\mathcal{N} - 1)$  equations that determine the independent auxiliary fields  $A_{\mu\nu}$ , which, in turn, determine the relative orientations of the Lorentz frames in (3.3). On multiplying by  $(g - A)_{\rho\alpha}(g - A)_{\sigma\beta}$  they take the compact form,

$$\left( L^\mu_\alpha(I) g_{\beta\nu} - L^\mu_\beta(I) g_{\alpha\nu} \right) \frac{\delta U}{\delta L^\mu_\nu(I)} = 0. \quad (4.7)$$

From the equivalence between the metric and the vielbein formulations it is clear that these equations must be equivalent to the  $6(\mathcal{N} - 1)$  constraints (2.6) obtained in [17]. To see this, note that, after the Lorentz frames of  $e^a_\mu(1)$  and  $\epsilon_{a_1 \dots a_d}$  are identified as in (3.5), the vielbeins  $e^a_\mu(I)$  are related to the  $E^a_\mu(I)$  of section 2 via

$$E^a_\mu(I) = \Lambda^a_b(1, I) e^b_\mu(I). \quad (4.8)$$

Then, converting the coordinate indices in (4.7) into Lorentz indices by multiplication with  $L^\alpha_\rho(I) L^\beta_\sigma(I) e^\rho_a(1) e^\sigma_b(1)$  and using  $\frac{\delta U}{\delta L^\mu_\nu} = \frac{\delta U}{\delta E^a_\rho} \frac{\delta E^a_\rho}{\delta \Lambda^c_d} \frac{\delta \Lambda^c_d}{\delta L^\mu_\nu}$  along with (4.8) and (3.11), we arrive at the constraint equations (2.6). In the following we exemplify the equivalence of the constraints in the two different formulations of the theory for two specific classes of interaction terms.

*Bimetric interactions:* In the multivielbein formulation, for interaction terms involving  $e^a_\mu(1)$  and only one other  $e^a_\mu(I)$ , the constraint (2.6) is simply solved by the condition (3.9) that expresses the vielbeins in terms of metrics. In the multimetric formulation this translates to the constraint (4.7) having a solution  $L^\mu_\nu(I) = \delta^\mu_\nu$ . We should therefore automatically find that (with  $L^{\mu\nu}(I) = L^\mu_\lambda(I) g^{\lambda\nu}(1)$ ),

$$\left( \frac{\delta U}{\delta L^{\mu\nu}} - \frac{\delta U}{\delta L^{\nu\mu}} \right) \Big|_{A=0} = 0. \quad (4.9)$$

This condition is indeed fulfilled in the metric description due to the symmetry of,

$$M_{\mu\nu}(I) \equiv g_{\mu\lambda}(1) \left[ \sqrt{g^{-1}(1)g(I)} \right]^\lambda_\nu, \quad (4.10)$$

in terms of which the mass potential in the bimetric theory has generic terms,

$$\sqrt{-\det g(1)} \tilde{\epsilon}^{\mu_1 \dots \mu_n \lambda_{n+1} \dots \lambda_d} \tilde{\epsilon}_{\nu_1 \dots \nu_n \lambda_{n+1} \dots \lambda_d} L^{\nu_1 \kappa_1}(I) M_{\kappa_1 \mu_1}(I) \dots L^{\nu_n \kappa_n}(I) M_{\kappa_n \mu_n}(I). \quad (4.11)$$

Differentiating this with respect to  $L^{\mu\nu}(I)$  and afterwards setting  $A(I) = 0$  or  $L^{\mu\nu}(I) = g^{\mu\nu}(1)$  results in a sum of terms proportional to matrices of the form,

$$M_\alpha^{\mu_1}(I) M_{\mu_1}^{\mu_2}(I) \dots M_{\mu_{n-1}}^\beta(I). \quad (4.12)$$

Each of these terms is manifestly symmetric by definition of  $M^\mu_\nu$ , which then directly implies (4.9). This also verifies that for  $\mathcal{N} = 2$  one recovers the bimetric theory [8] with the potential written in the notation of [24].

*Tri-metric interactions:* In order to provide both an explicit set-up in which  $A = 0$  is not a valid solution and another illustration of the equivalence to the symmetry conditions arising

from the vielbein formulation, we now consider the tri-metric term of [17] that includes  $e^a{}_\mu(1)$  and two other  $e^a{}_\mu(I)$ ,  $I = 2, 3$ . In terms of the metrics and with the definition (4.10), this term reads

$$\sqrt{-\det g(1)} \left( \text{Tr} \left[ L(2)M(2) \right] \text{Tr} \left[ L(3)M(3) \right] - \text{Tr} \left[ L(2)M(2)L(3)M(3) \right] \right). \quad (4.13)$$

In this case the 12 constraint equations (4.7) read

$$\begin{aligned} L_\alpha{}^\nu(2)M_{\nu\beta}(2)L^{\rho\sigma}(3)M_{\rho\sigma}(3) - L_\alpha{}^\nu(2)M_{\nu\rho}(2)L^{\rho\sigma}(3)M_{\sigma\beta}(3) - (\alpha \leftrightarrow \beta) &= 0, \\ L_\alpha{}^\nu(3)M_{\nu\beta}(3)L^{\rho\sigma}(2)M_{\rho\sigma}(2) - L_\alpha{}^\nu(3)M_{\nu\rho}(3)L^{\rho\sigma}(2)M_{\sigma\beta}(2) - (\alpha \leftrightarrow \beta) &= 0. \end{aligned} \quad (4.14)$$

Trying to enforce these conditions for  $A(I) = 0$ , i.e.  $L(I) = \mathbb{1}$ , we see that this would require  $M_{\sigma\alpha}(2)M^\alpha{}_\rho(3) = M_{\rho\alpha}(2)M^\alpha{}_\sigma(3)$ . Since the  $M(I)$  generically do not commute, we conclude that demanding  $A(I) = 0$  is not compatible with the symmetry constraints arising from the tri-vertex .

One can also verify that the conditions (4.14) are equivalent to the ones derived for the tri-vertex term in [17]. There, the conditions on the vielbeins  $E^a{}_\mu(I)$  read,

$$\begin{aligned} \left[ E^{-1}(1)E(2)\eta \text{Tr} \left( E^{-1}(1)E(3) \right) - E^{-1}(1)E(2)E^{-1}(1)E(3)\eta \right]_{ab} - (a \leftrightarrow b) &= 0, \\ \left[ E^{-1}(1)E(3)\eta \text{Tr} \left( E^{-1}(1)E(2) \right) - E^{-1}(1)E(3)E^{-1}(1)E(2)\eta \right]_{ab} - (a \leftrightarrow b) &= 0. \end{aligned} \quad (4.15)$$

Upon substituting  $L^\mu{}_L(I)M^\lambda{}_\nu(I) = E^\mu{}_a(1)E^a{}_\nu(I)$  into (4.14) and converting the Lorentz indices to spacetime indices by multiplication with  $e^\alpha{}_a(1)e^\beta{}_b(1)$ , we arrive at (4.15).

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